A Randomization Test for Autocorrelation

D. S. Robson

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Abstract

A test of the hypothesis that the sequence of random variables $\underset{\sim}{x} = (x_1, \ldots, x_n)$ are independent and identically distributed, against the alternative hypothesis that x_i and x_{i+1} are correlated, $i = 1, \ldots, n-1$, may be based on the statistic

$$T(\underset{\sim}{x}) = \frac{\sum_{i=1}^{n-1} (x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

with critical values determined from the randomization distribution of T. The mean and variance of this conditional distribution are given by

$$E\left\{T(\underset{\sim}{x}) \mid \{x_i\}\right\} = -\frac{1}{n}$$

$$\mathrm{Var}\left\{T(\underset{\sim}{x}) \mid \{x_i\}\right\} = \frac{n(n-1) + 1}{n^2(n-1)} - \frac{n+1}{n(n-1)} \frac{\sum_{i=1}^{n} (x_i - \bar{x})^4}{\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right]^2}$$

where $\{x_i\}$ denotes the unordered sample. If the vector $\underset{\sim}{x}$ is replaced by the vector $\underset{\sim}{R}$ of rank orders,

$$\underset{\sim}{R} = \left(R(x_1), \ldots, R(x_n)\right)$$

then the mean remains unchanged and the variance reduces to

$$\mathrm{Var}\left\{T(\underset{\sim}{R}) \mid \{R(x_i)\}\right\} = \frac{5n^3 - 19n^2 + 10n + 16}{5n^2(n-1)^2} .$$

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$$T(\underset{\sim}{x}) = \frac{\sum_{i=1}^{n-1}(x_i-\bar{x})(x_{i+1}-\bar{x})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

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$$E\left\{T(\underset{\sim}{x})\middle|\left\{x_i\right\}\right\} = -\frac{1}{n}$$

$$\operatorname{Var}\left\{T(\underset{\sim}{x})\middle|\left\{x_i\right\}\right\} = \frac{n(n-1)+1}{n^2(n-1)} - \frac{n+1}{n(n-1)}\,\frac{\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left[\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^2}$$

where $\{x_i\}$ denotes the unordered sample. If the vector $\underset{\sim}{x}$ is replaced by the vector $\underset{\sim}{R}$ of rank orders,

$$\underset{\sim}{R} = \left(R(x_1),\ldots,R(x_n)\right)$$

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$$\operatorname{Var}\left\{T(\underset{\sim}{R})\middle|\left\{R(x_i)\right\}\right\} = \frac{5n^3-19n^2+10n+16}{5n^2(n-1)^2}\,.$$

A complete enumeration of the 24 permutations for the case $n = 4$, for example, gives the following probability distribution for $T(\underset{\sim}{R})$:

$\underline{n = 4}$

Prob.	1/12	2/12	3/12	3/12	2/12	1/12
$T(\underset{\sim}{R}) + \frac{1}{4}$	-.5	-.4	-.1	+.1	+.4	+.5

The perfect symmetry appearing at $n = 4$ does not obtain for other sample sizes, however the distribution of $T(\underset{\sim}{R})$ does approach normality even for samples of modest size. At $n = 6$ the critical values

$$- \frac{1}{6} \pm 1.96 \frac{\sqrt{94.4}}{30}$$

isolate 16 sample configurations in the upper tail and 10 in the lower, as compared to the nominal .025(720) = 18. Those in the upper tail constitute 5 types:

1	2	3	4	5	6
1	2	3	4	6	5
2	1	3	5	6	4
2	1	3	4	6	5
1	2	3	5	6	4

and those in the lower tail are of three types:

3	6	1	5	2	4
3	5	1	6	2	4
3	4	2	6	1	5

(reversing the order does not alter the type).

The upper tail of positive autocorrelation tests for monotonicity in the sequence $x_1,\ldots,x_n$ while the lower tail of negative autocorrelation tests for cyclic oscillations. These two distinct alternatives to the null hypothesis could be tested separately, each in a variety of ways; monotonicity, for example, could be tested by the "peak test" or by the product moment correlation of either x_i or $R(x_i)$ with i, and the cyclic alternative is amenable to runs tests. Since the two-tailed autocorrelation test provides under one roof a single test against these two distinct alternatives there is good reason for operating at the 10% significance level with this procedure, thus providing a 5% test against each alternative.